have the following linear properties:

$$A(\alpha^1+\alpha^2)=A(\alpha^1)+A(\alpha^2), \lambda A(\alpha)=A(\lambda\alpha), \lambda\geqslant 0$$
$$B(\beta^1+\beta^2)=B(\beta^1)+B(\beta^2), \lambda B(\beta)=B(\lambda\beta), \lambda\geqslant 0 \tag{4.8}$$

From formulae (4.7) and (4.8) it follows that conditions (3.4) and (4.4) will be satisfied, if there are numbers $0<\lambda<1, \alpha_i\geqslant 0, \beta_i\geqslant 0$ such that

$$(1-\lambda)\langle\alpha_i\rangle_t^\tau=\lambda\alpha_i,\ \alpha_i\leqslant\langle\alpha_i\rangle_\tau^b,\ i=1,\ldots,k$$
$$\lambda\langle\beta_j\rangle_\tau^b=(1-\lambda)\beta_j,\ \beta_j\leqslant\langle\beta_j\rangle_t^\tau,\ j=1,\ldots,m$$

These conditions will be satisfied, if for all $i=1,\ldots,k$ and $j=1,\ldots,m$ we have

$$\langle\alpha_i\rangle_t^\tau\leqslant\lambda\langle\alpha_i\rangle_t^b,\ \lambda\langle\beta_j\rangle_t^b\leqslant\langle\beta_j\rangle_t^\tau,\ 0<\lambda<1$$

From here we obtain the condition which when satisfied results in the satisfaction of (4.3) and (4.4)

$$\max_i(\langle\alpha_i\rangle_t^\tau/\langle\alpha_i\rangle_t^b)\leqslant\min_j(\langle\beta_j\rangle_t^\tau/\langle\beta_j\rangle_t^b) \tag{4.9}$$

We shall now give some examples of multivalued functions that satisfy (4.8). If $A_1,\ldots,A_k$ are convex compacts in R^n, then $A(\alpha)=\alpha_1A_1+\ldots+\alpha_kA_k$ satisfies the condition (4.8).

Let $A_i, i=1,\ldots,n+1$ be defined by the scalar product of inequalities $(x_i,x)\leqslant 1$ in R^n. Here $x_1,\ldots,x_n,x_{n+1}$ are vectors from R^n and the first of them are linearly independent, and the coefficients f_i in expansion $x_{n+1}=f_1x_1+\ldots+f_nx_n$ are negative.

Consider the set

$$A(\alpha_1,\ldots,\alpha_{n+1})=\cap(\alpha_iA_i)=\{x\in R^n:(x_i,x)\leqslant\alpha_i, i=1,\ldots,n+1\} \tag{4.10}$$

in which α_i are non-negative. Then, as shown in /10/, the set (4.10) satisfies condition (4.8).

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SINGULAR PERTURBATIONS IN A CLASS OF PROBLEMS OF OPTIMAL CONTROL WITH INTEGRAL CONVEX CRITERION*

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The problem of optimal control is investigated with a linear law of motion and convex quality criterion. A small positive parameter appears in front of the derivatives of some of the unknowns in the law of motion. The behaviour of the optimal solution is studied when the small parameter approaches zero with some assumptions that are different from thos encountered in the literature.

1. Controlled objects whose law of motion is

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$$x^{\cdot} = A_{11}(t)\,x + A_{12}(t)\,y + B_1(t)\,u \tag{1.1}$$
$$\lambda y^{\cdot} = A_{21}(t)\,x + A_{22}(t)\,y + B_2(t)\,u;\quad \lambda \in (0,\ \Lambda_0),\ \Lambda_0 > 0$$

are usually called singularly perturbed /1,2/. It is assumed that the length of time $[t_0, T]$ is fixed, the phase vectors x and y belong to spaces R^n and R^m, respectively, the controlling parameter u belongs to space R^r, $A_{ij}(t)$, and $B_i(t)$, $(i, j = 1, 2)$ are matrices of corresponding dimensions that are continuous in the time interval $[t_0, T]$.

The behaviour of the solutions of various problems of the optimal control of an object with the law of motion (1.1), as $\lambda \to 0$ has been the subject of many publications (e.g. /3/, and surveys /1,2/). The basic assumption in these investigation is the stipulation that the real parts of the characteristic numbers of the matrix A_{22} must be negative. The case when these characteristic numbers have both positive and negative real parts was considered in /4/. Below we allow the real part of the characteristic numbers of the matrix A_{22} to vanish at one point θ_0 in the interval (t_0, T). The results can easily be transferred to the case when these real parts vanish at a finite number of points in the interval (t_0, T). The case is considered, when the real parts are non-positive, but similar results also hold, when the characteristic numbers of the matrix A_{22} may have non-positive and non-negative real parts. As an example of the problem of optimal control, the problem with a fixed right end is considered.

2. First, let us investigate some properties of the fundamental matrix $Y(t, \tau, \lambda)$, $\tau \leqslant t$, normalized for $t = \tau$ of the equation

$$\lambda y^{\cdot} = A_{22}(t)\,y \tag{2.1}$$

Let $\operatorname{Re}\gamma[A_{22}]$ be the real part of any characteristic number $\gamma[A_{22}]$ of the matrix A_{22}. Let us specify the assumptions that are assumed to be satisfied when studying the properties of the fundamental matrix $Y(t, \tau, \lambda)$.

A1. Points $\theta_0, \theta_1, \theta_2$, $t_0 < \theta_1 < \theta_0 < \theta_2 < T$ and a continuous function $\sigma(t)$, $t_0 \leqslant t \leqslant T$ exist which is linear in each segment $[t_0, \theta_1]$, $[\theta_1, \theta_0]$, $[\theta_0, \theta_2]$, $[\theta_2, T]$ and $\sigma(t) > 0$ when $t \neq \theta_0$ such that

$$\operatorname{Re}\gamma[A_{22}(t)] \leqslant -2\sigma(t)$$

also for some constant $\gamma_1 > 0$

$$\|\exp(A_{22}(t)\,\tau)\| \leqslant \gamma_1 \exp(-2\sigma(t)\,\tau),\quad \forall t \in [\theta_1,\ \theta_2],\ \tau \geqslant 0$$

A2. If $[\tau_1, \tau_2]$ is one of the segments $[\theta_1, \theta_0]$, $[\theta_0, \theta_2]$, and on that segment $\sigma(t) = at + b$, then

$$\|A_{22}(t_1) - A_{22}(t_2)\| \leqslant \gamma_0\,|t_1 - t_2|,\quad \forall t_1,\ t_2 \in [\tau_1,\ \tau_2];$$
$$\gamma_0 \in (0,\ 3\,|a|/\gamma_1)$$

Lemma 1. Let $[\tau_1, \tau_2]$ be any of the segments $[\theta_1, \theta_0]$, $[\theta_0, \theta_2]$, and the assumptions A1 and A2 are satisfied. Then a constant $c_0 > 0$ exists such that for all τ and t, $\tau_1 \leqslant \tau \leqslant t \leqslant \tau_2$, the inequality

$$\|Y(t,\ \tau,\ \lambda)\| \leqslant c_0 \exp(-(\sigma(t) + \sigma(\tau))(t - \tau)/(2\lambda)) \tag{2.2}$$

holds.

Proof. Let the function $\sigma(t) = at + b$ hold on the segment $[\tau_1, \tau_2]$. When $\theta \in [\tau_1, \tau_2]$, it follows from (2.1) that

$$\lambda dY(t,\tau,\lambda)/dt = A_{22}(\theta)\,Y(t,\tau,\lambda) + [A_{22}(t) - A_{22}(\theta)]\,Y(t,\tau,\lambda)$$

hence

$$Y(t,\tau,\lambda) = \exp\left(A_{22}(\theta)\frac{t-\tau}{\lambda}\right) + \frac{1}{\lambda}\int_\tau^t \exp\left(A_{22}(\theta)\,\frac{t-s}{\lambda}\right) \times [A_{22}(s) - A_{22}(\theta)]\,Y(s,\tau,\lambda)\,ds \tag{2.3}$$

First, we consider the case when the function $\sigma(t)$ decreases on the segment $[\tau_1, \tau_2]$ and $a < 0$. We assume that τ is a fixed point of the segment $[\tau_1, \tau_2]$ and that the notation

$$w_1(t,\tau,\lambda) = \|Y(t,\tau,\lambda)\|\,Z_1(t,\tau,\lambda)$$
$$Z_1(t,\tau,\lambda) = \exp\left((3\sigma(t) + \sigma(\tau))\frac{t-\tau}{2\lambda}\right),\quad M_1^{\lambda} = \max_{\tau_1 \leqslant \tau \leqslant t \leqslant \tau_2} w_1(t,\tau,\lambda)$$

is introduced.

If, when $\theta = \tau$ we multiply both sides of (2.3) by $Z_1(t,\tau,\lambda)$, when $\tau_1 \leqslant \tau \leqslant t \leqslant \tau_2$, we obtain

$$w_1(t,\tau,\lambda) \leqslant M_1^{\lambda} \leqslant \gamma_1(1 - \gamma_0\gamma_1/(3\,|a|))^{-1}$$

from which inequality (2.2) follows, when $c_0 = \gamma_1(1 - \gamma_0\gamma_1/(3\,|a|))^{-1}$.

When the function $\sigma(t) = at + b$ increases on the segment $[\tau_1, \tau_2]$ and $a > 0$, we introduce the notation

$$w_2(t, \tau, \lambda) = \| Y(t, \tau, \lambda) \| Z_2(t, \tau, \lambda)$$

$$Z_2(t, \tau, \lambda) = \exp\left((\sigma(t) + 3\sigma(\tau)) \frac{t-\tau}{2\lambda}\right), \quad M_2^{\tau} = \max_{\tau_1 \leqslant \tau \leqslant t \leqslant \tau_2} w_2(t, \tau, \lambda)$$

and consider the equation

$$\lambda dY(t, \tau, \lambda)/d\tau = -Y(t, \tau, \lambda) A_{22}(\tau)$$

from which we obtain, as above, the inequality (2.2) for the same value of c_0.

Lemma 2. Let the assumption A1 and A2 be satisfied. Then a constant $c_1 > 0$ exists such that for all fairly small $\lambda > 0$, if $[\tau_1, \tau_2]$ is any of segments $[t_0, \theta_1], [\theta_1, \theta_0], [\theta_0, \theta_2], [\theta_2, T]$, then

$$V_{\tau}^{\tau_2} \leqslant c_1, \quad V_{\tau_1}^{t} \leqslant c_1, \quad \forall \tau, \; t \in [\tau_1, \tau_2]$$

where $V_{\tau}^{\tau_2}$, $V_{\tau_1}^{t}$ is the total change of $Y(t, \tau, \lambda)$ with respect to t in the segment $[\tau, \tau_2]$, and with respect to τ in the segment $[\tau_1, t]$, respectively.

Proof. From assumption A1 in the segments $[t_0, \theta_0]$ and $[\theta_2, T]$ it follows that /5/

$$\| Y(t, \tau, \lambda) \| \leqslant c_0 \exp(-\sigma(t)(t-\tau)/\lambda)$$

where c_0 is some constant. The proof of the lemma for these segments is found, e.g. in /6/.

Next we consider the segment $[\theta_1, \theta_0]$. The uniform boundedness of $V_{\tau}^{\theta_0}$ with respect to $\tau \in [\theta_1, \theta_0]$, and $V_{\theta_1}^{t}$ with respect to $t \in [\theta_1, \theta_0]$ is obtained from the boundedness of the total change of the two terms on the right side of (2.3) when $\theta = \theta_1$. This follows for the first terms from the results in /6/, and for the second, from Lemma 1.

In the segment $[\theta_0, \theta_2]$ (2.3) is considered for $\theta = \theta_2$. This is followed by reasoning similar to that in the first case. This completes the proof of the lemma.

3. Let us study some of the properties of the solution $(\mathbf{x}_\lambda, \mathbf{y}_\lambda)$ of the problem

$$\mathbf{x}^{\boldsymbol{\cdot}} = A_{11}(t)\mathbf{x} + A_{12}(t)\mathbf{y} + \mathbf{f}(t, \lambda), \quad \mathbf{x}(t_0) = \mathbf{v}_0(\lambda) \tag{3.1}$$

$$\lambda \mathbf{y}^{\boldsymbol{\cdot}} = A_{21}(t)\mathbf{x} + A_{22}(t)\mathbf{y} + \mathbf{g}(t, \lambda), \quad \mathbf{y}(t_0) = \mathbf{w}_0(\lambda)$$

in the segment $[t_0, T]$ when $\lambda \in (0, \Lambda_0)$, assuming that matrix $A_{22}^{-1}(t)$ exists everywhere in the segment $[t_0, T]$. We denote by $\mathbf{x}_0$ the solution of the problem

$$\mathbf{x}^{\boldsymbol{\cdot}} = A_0(t)\mathbf{x} + \mathbf{f}(t, 0) - A_{12}(t) A_{22}^{-1}(t)\mathbf{g}(t, 0); \tag{3.2}$$

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

$$\mathbf{x}(t_0) = \mathbf{v}_0(0) - A_{12}(t_0) A_{22}^{-1}(t_0)\mathbf{w}_0$$

Let $X(t, \tau)$ be the fundamental matrix of the equation $\mathbf{x}^{\boldsymbol{\cdot}} = A_{11}(t)\mathbf{x}$ normalized for $t = \tau$, and $\mathbf{y}_0(t) = -A_{22}^{-1}(t)(A_{21}(t)\mathbf{x}_0(t) + \mathbf{g}(t, 0))$.

Theorem 1. Let the assumptions A1 and A2 be satisfied. Further, let us assume that when $\lambda \in [0, \Lambda_0)$ the functions $\mathbf{f}(\cdot, \lambda) \in L_p^{(n)}[t_0, T]$, $\mathbf{g}(\cdot, \lambda) \in L_p^{(m)}[t_0, T]$, $p > 1$; the set of functions $\mathbf{f}(\cdot, \lambda), \lambda \in (0, \Lambda_0)$ are bounded in $L_p^{(n)}[t_0, T]$, the set of points $\mathbf{v}_0(\lambda)$ and $\lambda \mathbf{w}_0(\lambda)$, $\lambda \in (0, \Lambda_0)$ are bounded in R^n and R^m, respectively. The following statements then hold.

1°. If the set of functions $\mathbf{g}(\cdot, \lambda), \lambda \in (0, \Lambda_0)$ is bounded in $L_p^{(m)}[t_0, T]$, a constant $c_2 > 0$ exists such that for all fairly small $\lambda > 0$

$$\max_{t_0 \leqslant t \leqslant T} \| \mathbf{x}_\lambda(t) \| \leqslant c_2$$

If the set of functions $\mathbf{g}(\cdot, \lambda), \lambda \in (0, \Lambda_0)$ is uniformly bounded, then for any point $\tau_0 \in (t_0, T)$ a constant $c_3 > 0$ exists that for all fairly small $\lambda > 0$

$$\max_{\tau_0 \leqslant t \leqslant T} \| \mathbf{y}_\lambda(t) \| \leqslant c_3 \tag{3.3}$$

If the set of points $\mathbf{w}_0(\lambda)$, $\lambda \in (0, \Lambda_0)$ is bounded, inequality (3.3) also holds for $\tau_0 = t_0$.

2°. If $\{\lambda_k\}_1^\infty$ is a sequence of numbers $\lambda_k > 0$ for which $\lim \lambda_k = 0$ as $k \to \infty$, the sequence $\{\mathbf{f}(\cdot, \lambda_k)\}_1^\infty$ is weakly convergent in $L_p^{(n)}[t_0, T]$ to $\mathbf{f}(\cdot, 0)$, and the sequence $\{\mathbf{g}(\lambda_k)\}_1^\infty$ weakly converges in $L_p^{(m)}[t_0, T]$ to $\mathbf{g}(\cdot, 0)$, $\lim \mathbf{v}_0(\lambda) = \mathbf{v}_0(0)$ and $\lim \lambda_k \mathbf{w}_0(\lambda_k) = \mathbf{w}_0$ as $k \to \infty$, then for any point $\tau_0 \in (t_0, T)$

$$\lim_{k \to \infty} \max_{\tau_0 \leqslant t \leqslant T} \| \mathbf{x}_{\lambda_k}(t) - \mathbf{x}_0(t) \| = 0,$$

and, if $\| \mathbf{w}_0 \| = 0$ the last equation also holds for $\tau_0 = t_0$.

3°. With the assumptions from 2°, when the functions $\mathbf{g}(\cdot, \lambda_k)(k = 1, 2, \ldots)$ are continuous and the sequences $\{g(\cdot, \lambda_k)\}_1^\infty$ converge uniformly in the segment $[t_0, T]$ to $g(\cdot, 0)$, then almost everywhere in the segment $[t_0,T]$ the sequence $\{y_{\lambda_k}(t)\}_1^\infty$ converges to $y_0(t)$.

The proof of all three statements of the theorem in the segment $[t_0, \theta_1]$ follows from Theorems 2.1 and 2.2 in /4/.

Let us now consider the segment $[\theta_1, \theta_0]$. We introduce the functions

$$v_1(t,\lambda) = X(t,\theta_1)(x_\lambda(\theta_1) - x_0(\theta_1)) + \int_{\theta_1}^{t} X(t,\tau)(f(\tau,\lambda) - f(\tau,0))\,d\tau + \int_{\theta_1}^{t} X(t,\tau)A_{12}(\tau)Y(\tau,\theta_1,\lambda)y_\lambda(\theta_1)\,d\tau - \left[\frac{1}{\lambda}\int_{\theta_1}^{t} X(t,\tau)A_{12}(\tau)\int_{\theta_1}^{\tau} Y(\tau,s,\lambda)A_{22}(s)y_0(s)\,ds\,d\tau + \int_{\theta_1}^{t} X(t,\tau)A_{12}(\tau)y_0(\tau)\,d\tau\right] + \frac{1}{\lambda}\int_{\theta_1}^{t} X(t,\tau)A_{12}(\tau) \times \int_{\theta_1}^{\tau} Y(\tau,s,\lambda)(g(s,\lambda) - g(s,0))\,ds\,d\tau$$

Since $\lim \lambda \| y_\lambda(\theta_1) \| = 0$ as $\lambda \to 0$, the uniform boundedness of the set of functions $v_1(\cdot, \lambda)$ for all fairly small $\lambda > 0$ on assumptions 1° is proved directly. For the last three terms we use the properties of the Stieltjes integral and Lemma 2.

According to (3.1) and (3.2)

$$\| x_\lambda(t) - x_0(t) \| \leqslant \max_{\theta_1 \leqslant \tau \leqslant \theta_0} \| v_1(\tau,\lambda) \| + \left\| \frac{1}{\lambda}\int_{\theta_1}^{t} X(t,\tau)A_{12}(\tau)\int_{\theta_1}^{\tau} Y(\tau,s,\lambda)A_{21}(s)(x_\lambda(s) - x_0(s))\,ds\,d\tau \right\|$$

Then, changing the order of integration and applying the Gronwall inequality, we obtain

$$\| x_\lambda(t) - x_0(t) \| \leqslant \max_{\theta_1 \leqslant \tau \leqslant \theta_0} \| v_1(\tau,\lambda) \| \exp\Big(\max_{\theta_1 \leqslant s \leqslant z \leqslant \theta_0} \Big\| \int_{s}^{z} X(t,\tau)A_{12}(\tau)A_{22}^{-1}\,dY(\tau,s,\lambda) \Big\| \times \int_{\theta_1}^{t} \| A_{21}(s) \|\,ds \Big) \tag{3.4}$$

Consequently, by virtue of the assumptions in 1° the set of functions x_λ is uniformly bounded for all fairly small $\lambda > 0$. On the same assumptions we have

$$\lim_{\lambda \to 0} \lambda \| y_\lambda(\theta_0) \| \leqslant + \lim_{\lambda \to 0} \| \lambda Y(\theta_0,\theta_1,\lambda) y_\lambda(\theta_1) \| + \left\| \int_{\theta_1}^{\theta_0} Y(\theta_0,\tau,\lambda)A_{21}(\tau)x_\lambda(\tau)\,d\tau \right\| + \Big(\int_{\theta_1}^{\theta_0} \| Y(\theta_0,\tau,\lambda) \|^q\,d\tau \Big)^{1/q} \Big(\int_{\theta_1}^{\theta_0} \| g(r,\lambda_k) \|^p\,d\tau \Big)^{1/p} = 0, \quad \frac{1}{p} + \frac{1}{q} = 1$$

When the set of functions $g(\cdot, \lambda)$, $\lambda \in (0, \Lambda_0)$ is uniformly bounded, from the equation

$$y_\lambda(t) = Y(t,\theta_1,\lambda)y_\lambda(\theta_1) + \frac{1}{\lambda}\int_{\theta_1}^{t} Y(t,\tau,\lambda)A_{21}(\tau)x_\lambda(\tau)\,d\tau + \frac{1}{\lambda}\int_{\theta_1}^{t} Y(t,\tau,\lambda)g(\tau,\lambda)\,d\tau \tag{3.5}$$

there follows the uniform boundedness of the set of functions y_λ for all fairly small $\lambda > 0$.

In assumptions 2° the sequence $\{v_1(\cdot, \lambda_k)\}_1^\infty$ converges uniformly in the segment $[\theta_1, \theta_0]$ to zero, and then from (3.4) there follows the uniform convergence of the sequence $\{x_{\lambda_k}\}_1^\infty$ to x_0. The convergence of the sequence $\{y_{\lambda_k}\}_1^\infty$ for all $t \in (\theta_1, \theta_0]$ on assumptions 3° is obtained from (3.5) using Lemmas 1, 2 and the properties of the Stieltjes integral.

Let us consider in the segment $[\theta_0, \theta_2]$ the functions $v_2(t, \lambda)$ obtained from $v_1(t, \lambda)$ by exchanging θ_1 for θ_0. Similarly, we shall prove that under appropriate conditions on assumption 1° for all fairly small $\lambda > 0$ the set of functions x_λ is uniformly bounded $\lim \lambda \| y_\lambda(\theta_2) \| = 0$ as $\lambda \to 0$ and the set of functions y_λ is uniformly bounded. The uniform convergence in the segment $[\theta_0, \theta_2]$ of the sequence $\{x_{\lambda_k}\}_1^\infty$ to x_0 and the convergence of the sequence $\{y_{\lambda_k}(t)\}_1^\infty$ to $y_0(t)$ for all $t \in (\theta_0, \theta_2]$ is similarly proved. The proof of the theorem is completed by applying in the segment $[\theta_2, T]$. Theorems 2.1 and 2.2 from /4/.

4. Let us investigate some of the properties of the controlled object whose law of motion is (1.1). The proof of all statements cited below is similar to that of the respective statements in /4/, and is carried out using the scheme employed there but taking Theorem 1 into account.

Let $f^0(t, \mathbf{x})$ be a scalar function continuous in the set $[t_0, T] \times R^n$, convex in $\mathbf{x}$ for any fixed value $t \in [t_0, T]$, $f^0(t, \mathbf{x}) \geqslant 0$, and have continuous partial derivatives $\partial f^0(t, \mathbf{x})/\partial \mathbf{x}$. Let $h(t, \mathbf{u})$ be a scalar function continuous in the set $[t_0, T] \times R^r$ strictly convex in $\mathbf{u}$ for fixed $t \in [t_0, T]$, and for some constants $a_0 > 0$, $p > 1$ the inequality $h(t, \mathbf{u}) \geqslant a_0 \|\mathbf{u}\|^p$ is satisfied. The admissable control for $\lambda \in [0, \Lambda_0)$ comprises all r-measurable functions $\mathbf{u} \in L_p^{(r)}[t_0, T]$ for which the functional

$$I(\mathbf{u}, \lambda) = \int_{t_0}^{T} \{f^0(t, \mathbf{x}(t)) + h(t, \mathbf{u}(t))\}\, dt \tag{4.1}$$

where $(\mathbf{x}, \mathbf{y})$ is the solution of system (1.1) corresponding to the control $\mathbf{u}$, takes a finite value.

The following are the assumptions for which investigations are carried out below.

B1. The equation

$$\operatorname{rank}[B_2(T)\, A_{22}(T)\, B_2(T) \ldots A_{22}^{m-1}(T)\, B_2(T)] = m$$

holds

B2. An object whose law of motion is

$$\mathbf{x}^{\cdot} = A_0(t)\,\mathbf{x} + (B_1(t) - A_{12}(t)\, A_{22}^{-1}(t)\, B_2(t))\,\mathbf{u} \tag{4.2}$$

is entirely controllable in the segment $[t_0, T]$.

First, we formulate a lemma which will be repeatedly used in proving the following theorems.

Lemma 3. Let the assumptions A1, A2 and B1 be satisfied. Further, let $\{\lambda_k\}_1^\infty$ be a sequence of numbers $\lambda_k > 0, \lim \lambda_k = 0$ when $k \to \infty$; $\mathbf{w}_0 \in R^m$, $\mathbf{w}_T \in R^m$; let $\mathbf{u}^*(t)$, $t_0 \leqslant t \leqslant T$ be a continuous control with the corresponding solution $\mathbf{x}^*$ of (4.2). Then, a sequence $\{\mathbf{u}_k^*\}_1^\infty$ of r-measurable functions $\mathbf{u}_k^*$ exists with the respective solution $(\mathbf{x}_k^*, \mathbf{y}_k^*)$ of system (1.1) when $\lambda = \lambda_k$, so that

1°. the sequences $\{\mathbf{u}^*_k\}_1^\infty$, $\{\mathbf{x}_k^*\}_1^\infty$, $\{\mathbf{y}_k^*\}_1^\infty$ are uniformly bound in the segment $[t_0, T]$, and at each point $t \in (t_0, T)$, $t \neq \theta_0$ converge to $\mathbf{u}^*$, $\mathbf{x}^*$, $\mathbf{y}^* = -A_{22}^{-1}(A_{21}\mathbf{x}^* + B_2\mathbf{u}^*)$, respectively

2°. the equations

$$\mathbf{x}_k^*(t_0) = \mathbf{x}^*(t_0),\ \lim_{k\to\infty} \mathbf{x}_k^*(T) = \mathbf{x}^*(T),\quad \mathbf{y}_k^*(t_0) = \mathbf{w}_0,\ \mathbf{y}_k^*(T) = \mathbf{w}_T$$

$$\lim_{k\to\infty} I(\mathbf{u}_k^*, \lambda_k) = I(\mathbf{u}^*, 0)$$

hold.

If also assumptions B2 holds, the sequence $\{\mathbf{u}_k^*\}_1^\infty$ may be selected so that the equation $\mathbf{x}_k^*(T) = \mathbf{x}^*(T)$ is satisfied.

Theorem 2. Let the assumptions A1, A2, B1 and B2 be satisfied. Then, for all fairly small $\lambda > 0$ the object whose law of motion is (1.1) is entirely controllable in the segment $[t_0, T]$.

Suppose P_λ, $\lambda \in (0, \Lambda_0)$ denotes the problem of optimal control of an object whose law of motion is (1.1), which consists of finding an admissable control $\mathbf{u}$ which transfers the object from the initial state $\mathbf{x}(t_0) = \mathbf{v}_0$, $\mathbf{y}(t_0) = \mathbf{w}_0$ to the final state $\mathbf{x}(T) = \mathbf{v}_T$, $\mathbf{y}(T) = \mathbf{w}_T$ for the minimal value of the criterion (4.1). We denote the optimal control corresponding to the solution of system (1.1) and the optimal value of the criterion (4.1) for problem P_λ by $\mathbf{u}_\lambda$, $(\mathbf{x}_\lambda, \mathbf{y}_\lambda)$, I_λ, respectively. We denote by P_0 the problem of optimal control which consists of finding an admissible control $\mathbf{u}$ with corresponding solution $\mathbf{x}$ of (4.2) such that $\mathbf{x}(t_0) = \mathbf{v}_0$, $\mathbf{x}(T) = \mathbf{v}_T$ and the criterion (4.1) takes the minimum value. Let $\mathbf{u}_0$, $\mathbf{x}_0$, I_0 be the solution of this problem and $\mathbf{y}_0 = -A_{22}^{-1}(A_{21}\mathbf{x}_0 + B_2\mathbf{u}_0)$.

Theorm 3. Let assumptions A1, A2, B1, and B2 be satisfied. Then for any number $\varepsilon > 0$ and any four points $\tau_0^0, \tau_1^0, \tau_2^0, \tau_3^0$ for which

$$t_0 < \tau_0^\circ < \theta_1 < \tau_1^\circ < \theta_0 < \tau_2^\circ < \theta_2 < \tau_3^\circ < T$$

a number $\delta > 0$ exists such that, when $\lambda \in (0, \delta)$, then

$$|I_\lambda - I_0| + \max_{t_0 \leqslant t \leqslant T} \|\mathbf{x}_\lambda(t) - \mathbf{x}_0(t)\| + \max_{t_0 \leqslant t \leqslant \tau_3^\circ} \|\mathbf{u}_\lambda(t) - \mathbf{u}_0(t)\| + \max_{t \in [\tau_0^\circ, \tau_3^\circ] \setminus [\tau_1^\circ, \tau_2^\circ]} \|\mathbf{y}_\lambda(t) - \mathbf{y}_0(t)\| < \varepsilon$$

Note that using the statements of Sects. 2 and 3 enables us to extend the results of Theorems 2 and 3 to the case when the characteristic numbers of the matrix A_{22} in (1.1) have real parts that change signs at one point of the interval (t_0, T).

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ON THE DEFINITION OF VARIATIONS IN THE MECHANICS OF CONTINUOUS MEDIA*

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The basic forms of variations used in the mechanics of continuous media are presented, and relations between various types of variations of vectors and tensors are established.

The construction of new more complex models of continuous media can be based on the use of the variational equation /1/. In constructing models of continuous dislocations of plastic and solid media interacting with an electromagnetic field (in Newtonian mechanics as well as in the theory of relativity) /2-6/ and, also, a number of other models, it is necessary to deal with variations of various types of different quantities, such as scalars, vectors, and tensors which can be considered as functions of Euler or Lagrangian coordinates. Hence it is necessary to have established connections between various types of variations which are of the same nature as the variable functions.

Below we consider some of the simplest types of variations used to construct models of solid media in the special theory of relativity. We shall denote by x^i $(i = 1, 2, 3, 4)$ the Euler coordinates and by ξ^a $(a = 1, 2, 3, 4)$ the Lagrangian coordinates of four-dimensional Minkowski space, assuming that the global, coordinates x^4 and ξ^4 have a temporal nature $x^4 = ct$, $\xi^4 = c\tau$, (c is the velocity of light in a vacuum).

In the coordinate system x^i with basis vectors $\mathbf{э}_i$ defined as unit vectors tangent to the lines $x^i = \text{const}$, and the particle world lines determined by the equations $x^i = x^i(\xi^a)$ (the law of motion of a point with Lagrangian coordinates relative to system x^i). Here and henceforth Greek indices run through the numbers 1,2,3, and the lower case Latin letters through 1,2,3,4.

At each point of the Minkowski four-dimensional space-time we may introduce covariant and contravariant basis vectors ($\mathbf{э}_i$, and $\mathbf{э}^i$, $\mathbf{э}_a{}^\wedge$ and $\mathbf{э}^{\wedge a}$ for coordinates x^i and for systems ξ^α, respectively, connected by the equations

$$\mathbf{э}_a{}^\wedge = \frac{\partial x^i}{\partial \xi^a}\mathbf{э}_i = x_a{}^i \mathbf{э}_i, \quad \mathbf{э}_i = \frac{\partial \xi^a}{\partial x^i} = \xi_i{}^a \mathbf{э}_a{}^\wedge$$

When constructing models of media and fields besides the law of motion one has to consider various scalar, vector, and tensor fields that represent mechanical, physical, or chemical characteristic of the phenomena and processes investigated which are functions of the coordinates x^i or ξ^a (for details of these characteristics see, e.g., /6/). In problems related to specifying or determining the laws of motion of the solid medium, and the laws of variation

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